COMMUTATOR IDEALS OF SUBALGEBRAS OF TOEPLITZ ALGEBRAS ON WEIGHTED BERGMAN SPACES

TRIEU LE

Communicated by Kenneth R. Davidson

ABSTRACT. For $\alpha > -1$, let $A^2_\alpha$ denote the corresponding weighted Bergman space of the unit ball. For any self-adjoint subset $G \subset L^\infty$, let $T(G)$ denote the $C^*$-subalgebra of $\mathcal{B}(A^2_\alpha)$ generated by $\{T_f : f \in G\}$. Let $CT(G)$ denote the commutator ideal of $T(G)$. It was showed by D. Suárez (in 2004 for $n = 1$) and by the author (in 2006 for all $n \geq 1$) that $CT(L^\infty) = T(L^\infty)$ in the case $\alpha = 0$. In this paper we show that in the setting of weighted Bergman spaces, the identity $CT(G) = T(G)$ holds true for a class of subsets $G$ including $L^\infty$.

KEYWORDS: Commutator ideals; Toeplitz operators; Weighted Bergman spaces.

MSC (2000): Primary 47B35; Secondary 47B47.

1. INTRODUCTION

For any integer $n \geq 1$, let $\mathbb{C}^n$ denote the Cartesian product of $n$ copies of $\mathbb{C}$. For any $z = (z_1, \ldots, z_n)$ and $w = (w_1, \ldots, w_n)$ in $\mathbb{C}^n$, we write $(z, w) = z_1\bar{w}_1 + \cdots + z_n\bar{w}_n$ and $|z| = \sqrt{|z_1|^2 + \cdots + |z_n|^2}$ for the inner product and the associated Euclidean norm. Let $B_n$ denote the open unit ball which consists of all $z \in \mathbb{C}^n$ with $|z| < 1$. Let $S_n$ denote the unit sphere which consists of all $z \in \mathbb{C}^n$ with $|z| = 1$. We write $\overline{B}_n$ for the closed unit ball which is $B_n \cup S_n$. Let $C(\overline{B}_n)$ (respectively, $C(S_n)$) denote the space of all functions that are continuous in the open unit ball (respectively, the closed unit ball).

Let $\nu$ denote the Lebesgue measure on $\overline{B}_n$ normalized so that $\nu(\overline{B}_n) = 1$. Fix a real number $\alpha > -1$. The weighted Lebesgue measure $\nu_{\alpha}$ on $\overline{B}_n$ is defined by $\nu_{\alpha}(z) = c_\alpha (1 - |z|^2)^{\alpha} \nu(z)$, where $c_\alpha$ is a normalizing constant so that $\nu_{\alpha}(\overline{B}_n) = 1$. A direct computation shows that $c_\alpha = \frac{\Gamma(n + \alpha + 1)}{\Gamma(n + 1)\Gamma(\alpha + 1)}$. For $1 \leq p \leq \infty$, let $L^p_{\alpha}(\overline{B}_n)$ denote the space $L^p(\overline{B}_n, d\nu_{\alpha})$. Note that $L^\infty_{\alpha}$ is the same as $L^\infty(\overline{B}_n, d\nu)$. 
For any $f$ in $L^p_\alpha$, we denote by $\|f\|_{p,\alpha}$ the norm of $f$. So

$$\|f\|_{p,\alpha} = \left( \int_{\mathbb{B}_n} |f(z)|^p \, dv_\alpha \right)^{1/p}.$$ 

The weighted Bergman space $A^p_\alpha$ consists of all holomorphic functions in $\mathbb{B}_n$ which are also in $L^p_\alpha$. It is well-known that $A^p_\alpha$ is a closed subspace of $L^p_\alpha$. In this paper we are only interested in the case $p = 2$. We denote the inner product in $L^2_\alpha$ by $\langle \cdot, \cdot \rangle_\alpha$. Since $A^2_\alpha$ is a closed subspace of the Hilbert space $L^2_\alpha$, there is an orthogonal projection $P_\alpha$ from $L^2_\alpha$ onto $A^2_\alpha$. For any $f \in L^2_\alpha$, we have

$$(P_\alpha f)(z) = \int_{\mathbb{B}_n} \frac{f(w)}{(1 - \langle z, w \rangle)^{n+1+\alpha}} \, dv_\alpha(w) = \langle f, K^2_\alpha \rangle_\alpha, \quad z \in \mathbb{B}_n,$$

where $K^2_\alpha(w) = \frac{1}{(1 - \langle w, z \rangle)^{n+1+\alpha}}$. The functions $K^2_\alpha$'s are the reproducing kernels for $A^2_\alpha$. For any $z \in \mathbb{B}_n$, let $k^2_\alpha(w) = \frac{K^2_\alpha(w)}{\|K^2_\alpha\|_{2,\alpha}}$. Then $\|k^2_\alpha\|_{2,\alpha} = 1$ and

$$\langle g, k^2_\alpha \rangle_\alpha = (1 - |z|^2)^{(n+1+\alpha)/2} g(z)$$

for $g \in A^2_\alpha$ and $z \in \mathbb{B}_n$.

For any multi-index $m = (m_1, \ldots, m_n)$ of non-negative integers, we write $|m| = m_1 + \cdots + m_n$ and $m! = m_1! \cdots m_n!$. For any $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, we write $z^m = z_1^{m_1} \cdots z_n^{m_n}$. The standard orthonormal basis for $A^2_\alpha$ is $\{ e_m : m \in \mathbb{N}^n \}$, where

$$e_m(z) = \left[ \frac{n(m + \alpha + 1)}{m! (n + \alpha + 1)} \right]^{1/2} z^m, \quad m \in \mathbb{N}^n, z \in \mathbb{B}_n.$$ 

For a more detailed discussion of $A^2_\alpha$, see Chapter 2 in [7].

For any $f \in L^\infty$, the Toeplitz operator $T_f : A^2_\alpha \to A^2_\alpha$ is defined by the formula $T_fh = P_\alpha(fh)$ for $h \in A^2_\alpha$. We have

$$T_fh(z) = \int_{\mathbb{B}_n} \frac{f(w)h(w)}{(1 - \langle z, w \rangle)^{n+1+\alpha}} \, dv_\alpha(w)$$

for $h \in A^2_\alpha$ and $z \in \mathbb{B}_n$.

We have $T_f^* = T_f$ and since $\|P_\alpha\| = 1$, $\|T_f\| \leq \|f\|_{\infty}$ for all $f \in L^\infty$. There are examples of $f \in L^\infty$ so that $\|T_f\| < \|f\|_{\infty}$. Since $T_f$ is an integral operator given by equation (1.1), we see that $T_f$ is compact if $f$ vanishes on the complement of a compact subset of $\mathbb{B}_n$.

Let $\mathfrak{B}(A^2_\alpha)$ denote the $C^*$-algebra of all bounded linear operators on $A^2_\alpha$. Let $\mathcal{K}$ denote the ideal of $\mathfrak{B}(A^2_\alpha)$ that consists of all compact operators. For any subset $G$ of $L^\infty$, let $\mathfrak{F}(G)$ denote the Banach subalgebra of $\mathfrak{B}(A^2_\alpha)$ generated by all $T_f$ with $f \in G$ and let $\mathfrak{F}^\circ(G)$ denote the closed two-sided ideal of $\mathfrak{F}(G)$ generated by all commutators $[T_f, T_g] = T_fT_g - T_gT_f$ with $f, g \in G$. When $G$ is
self-adjoint (i.e. $G = G^{*} = \{f : f \in G\}$), $\mathfrak{T}(G)$ is a $C^*$-algebra and so the quotient $\mathfrak{T}(G)/\mathfrak{cT}(G)$ is also a $C^*$-algebra. The algebra $\mathfrak{T} = \mathfrak{T}(L^\infty)$ is called the full Toeplitz algebra. Its commutator ideal is denoted by $\mathfrak{cT}$.

It is well-known that $\mathfrak{cT}(C(B_n)) = \mathcal{K}$. This was showed by L.A. Coburn [1] for the unweighted Bergman space ($\alpha = 0$) but his argument can be easily adapted to the weighted cases. In the unweighted case, D. Suárez [6] showed that $\mathfrak{cT}(G) = \mathfrak{T}$ for $n = 1$. This result was generalized by the author to all $n \geq 1$ in [2]. The purpose of this paper is to show that the identity $\mathfrak{cT}(G) = \mathfrak{T}(G)$ holds true in the general setting of weighted Bergman spaces for a class of self-adjoint subsets $G$ of $L^\infty$. In the current situation, even though explicit computations used by Suárez cannot be used, many of his results play an important role. In particular, Proposition 2.3 is a version of Proposition 2.9 in [6] for Toeplitz operators on weighted Bergman spaces.

We next introduce a metric on $B_n$ which we will mainly use in the paper. For $z \in B_n$, let $\varphi_z$ denote the Möbius automorphism of $B_n$ that interchanges $0$ and $z$. For any $w \in B_n$, let $\rho(z, w) = |\varphi_z(w)|$. Then $\rho$ is a metric on $B_n$ called the pseudo-hyperbolic metric which is invariant under the action of the automorphism group $\text{Aut}(B_n)$ of $B_n$. For any $z, w, u \in B_n$, we have

$$\rho(z, w) \geq \rho(z, u) - \rho(u, w) \frac{\rho(z, u) - \rho(u, w)}{1 - \rho(z, u)p(u, w)}.$$  

See Section 2 in [2] for more details about the metric $\rho$.

For any $a$ in $B_n$ and any $0 < r < 1$, let $E(a, r) = \{z \in B_n : \rho(z, a) < r\}$, which is the ball of radius $r$ centered at $a$ with respect to the metric $\rho$.

A sequence $\{z_j : j \in J\} \subset B_n$ ($J$ is either a non-empty finite set or $J = \mathbb{N}$) is said to be separated if $r = \inf\{\rho(z_{j_1}, z_{j_2}) : j_1, j_2 \in J, j_1 \neq j_2\} > 0$. The number $r$ is called the degree of separation of the sequence.

The following theorem is the main result of the paper.

**Theorem 1.1.** Suppose $W \subset B_n$ is a measurable set and suppose there exist real numbers $r, R \in (0, 1)$ and a separated sequence $\{z_j : j \in J\} \subset B_n$ ($J$ is either a non-empty finite set or $J = \mathbb{N}$) so that

$$\bigcup_{j \in J} E(z_j, r) \subset W \subset \bigcup_{j \in J} E(z_j, R).$$

Let $G_1(W) = \{f \in C(B_n) : f \text{ vanishes on } B_n \setminus W\}$ and $G_2(W) = \{f \in L^\infty : f \text{ vanishes a.e. on } B_n \setminus W\} = \chi_W L^\infty$. Let $G$ be a self-adjoint linear subspace of $G_2(W)$ that contains $G_1(W)$. Then if $J$ is finite, $\mathcal{K} = \mathfrak{cT}(G) = \mathfrak{T}(G)$. If $J = \mathbb{N}$, we have $\mathcal{K} \subsetneq \mathfrak{cT}(G) = \mathfrak{T}(G)$.

Applying Theorem 1.1 with $G = G_2(W)$, we conclude that for any measurable set $W \subset B_n$ satisfying (1.3),

$$\mathfrak{cT}(\chi_W L^\infty) = \mathfrak{T}(\chi_W L^\infty).$$
Now by an argument using Zorn’s Lemma, we see that for any $0 < R < 1$ there is a separated sequence $\{z_j : j \in \mathbb{N}\}$ of $\mathbb{B}_n$ so that $\mathbb{B}_n = \bigcup_{j \in \mathbb{N}} E(z_j, R)$. We then get $\mathcal{C} \Sigma = \Sigma$. This identity in the case of unweighted Bergman space is our earlier result in [2].

Note that (1.4) holds true in many cases where $W$ need not satisfy (1.3). In fact, if $W$ is any measurable subset of $\mathbb{B}_n$ so that $T \chi W$ is a compact operator then $\Sigma(\chi W L^\infty) \subset K$, and so $K = \mathcal{C} \Sigma(\chi W L^\infty) = \Sigma(\chi W L^\infty)$. Thus (1.3) is not a necessary condition for (1.4) to hold true. It would be interesting to know an explicit description of those $W$’s that give (1.4) but at this time we do not know if there exists such a description.

The rest of the paper is organized as follows. In Section 2, we provides some preliminaries and basic results. In Section 3, we give a proof for Theorem 1.1.

2. BASIC RESULTS

In the first part of this section, we discuss how some results of Suárez can be generalized to the case of weighted Bergman space $A^2_\alpha$. These results play an important role in the proof of Theorem 1.1. In the second part of the section, we show that a Toeplitz operator with radial symbol (see definition below) is a diagonal operator with respect to the standard orthonormal basis of $A^2_\alpha$. A formula for the eigenvalues of the operator is given. We also show that the $\mathcal{C}^*$-algebra $\mathcal{T}(\mathcal{C}(\mathbb{B}_n))$ with $f$ a radial function is irreducible.

**Lemma 2.1.** Let $0 < r < R < 1$ and $\{z_j\}_{j=1}^\infty \subset \mathbb{B}_n$ such that $E(z_{j_1}, r) \cap E(z_{j_2}, r) = \emptyset$ if $j_1 \neq j_2$. Let

$$
\Phi(z, w) = \sum_{j=1}^\infty \chi_{E(z_j, R)}(z) \chi_{\mathbb{B}_n \setminus E(z_j, R)}(w) |1 - \langle z, w \rangle|^{-n-1-\alpha}.
$$

Then for any $0 < \beta < \alpha + 1$, we have

(i) $\int_{\mathbb{B}_n} \Phi(z, w) (1 - |w|^2)^{-\beta} \, dv(z) \leq C_1 (1 - |z|^2)^{-\beta}$ for all $z \in \mathbb{B}_n$;

(ii) $\int_{\mathbb{B}_n} \Phi(z, w) (1 - |z|^2)^{-\beta} \, dv(z) \leq C_2 (1 - |w|^2)^{-\beta}$ for all $w \in \mathbb{B}_n$;

where $C_1 = C_1(n, \alpha, \beta)$, $C_2 = C_2(n, \alpha, \beta, r, R)$ and $C_2 \to 0$ as $R \uparrow 1$. 
Proof. For any $z, w \in \mathbb{B}_n$, $\Phi(z, w) \leq |1 - (z, w)|^{-n-1-a}$, so

$$\int_{\mathbb{B}_n} \Phi(z, w) (1 - |w|^2)^{-\beta} \, d\nu_a(z) \leq \int_{\mathbb{B}_n} \frac{(1 - |w|^2)^{-\beta}}{|1 - (z, w)|^{n+1+a}} \, d\nu_a(w)$$

$$= c_a \int_{\mathbb{B}_n} \frac{(1 - |w|^2)^{n-\beta}}{|1 - (z, w)|^{n+1+a}} \, d\nu(w).$$

Applying Theorem 1.12 in [7] with $t = \alpha - \beta > -1$ and $c = \beta > 0$, we get a constant $C_1 = C_1(n, \alpha, \beta)$ so that the inequality in (i) holds.

Now fix $w \in \mathbb{B}_n$. For any $z \in \mathbb{B}_n$, if $\Phi(z, w) \neq 0$ then there is an integer $j \geq 1$ so that $z \in E(z_j, r)$ and $w \notin E(z_j, R)$. Then

$$\rho(z, w) \geq \frac{\rho(w, z_j) - \rho(z, z_j)}{1 - \rho(w, z_j)} \geq \frac{R - r}{1 - Rr}.$$ 

Let $\delta = \frac{R - r}{1 - Rr}$. It is clear that for each $0 < r < 1, \delta \uparrow 1$ as $R \uparrow 1$. We have

$$\int_{\mathbb{B}_n} \Phi(z, w) (1 - |z|^2)^{-\beta} \, d\nu_a(z)$$

$$\leq \int_{\mathbb{B}_n \setminus E(w, \delta)} \Phi(z, w) (1 - |z|^2)^{-\beta} \, d\nu_a(z)$$

$$\leq \int_{\mathbb{B}_n \setminus E(w, \delta)} \frac{(1 - |z|^2)^{-\beta}}{|1 - (z, w)|^{n+1+a}} \, d\nu_a(z)$$

$$= \int_{|\eta| > \delta} \frac{(1 - |\varphi_w(\eta)|^2)^{-\beta}}{|1 - (\varphi_w(\eta), w)|^{n+1+a}} \, d\nu_a(\eta)$$

(by the change-of-variable $z = \varphi_w(\eta)$ - see Proposition 1.13 in [7])

$$= \int_{|\eta| > \delta} \frac{(1 - |w|^2)^{n-\beta}(1 - |\eta|^2)^{-\beta}}{|1 - (w, \eta)|^{n+1+a-2\beta}} \, d\nu_a(\eta)$$

$$= c_a (1 - |w|^2)^{-\beta} \int_{|\eta| > \delta} \frac{(1 - |\eta|^2)^{n-\beta}}{|1 - (w, \eta)|^{n+1+a-2\beta}} \, d\nu(\eta).$$

Here in the fourth equality, we have used the formulas

$$1 - |\varphi_w(\eta)|^2 = \frac{(1 - |w|^2)(1 - |\eta|^2)}{|1 - (w, \eta)|^2} \quad \text{and} \quad 1 - (\varphi_w(\eta), w) = \frac{1 - |w|^2}{1 - (\eta, w)}.$$ 

See Theorem 2.2.2 in [4] or Lemma 1.3 in [7].
For any $1 < \gamma < \infty$, Holder’s inequality gives
\[
\int_{|\eta| > \delta} \frac{(1 - |\eta|^2)^{a - \beta}}{|1 - \langle w, \eta \rangle|^{\gamma(n + 1 + a - 2\beta)\gamma}} d\nu(\eta)
\leq \left( \int_{|\eta| > \delta} \frac{(1 - |\eta|^2)^{(a - \beta)\gamma}}{|1 - \langle w, \eta \rangle|^{\gamma(n + 1 + a - 2\beta)\gamma}} d\nu(\eta) \right)^{1/\gamma} \left(1 - \delta^{2n}\right)^{1 - 1/\gamma}.
\]

If $a - \beta \geq 0$, choose any $\gamma > \gamma(n, a, \beta) > 1$ so that $(n + 1 - \beta)\gamma - (n + 1) < 0$ (since $n + 1 - \beta < n + 1$, it is possible to choose such a $\gamma$). If $a - \beta < 0$, choose any $\gamma = \gamma(n, a, \beta) > 1$ so that $\gamma > \frac{1}{\beta - a}$ and $(n + 1 - \beta)\gamma - (n + 1) < 0$ (since $0 < \beta - a < 1$ and $n + 1 - \beta < n + 1$, it is possible to choose such a $\gamma$). With the above choice of $\gamma$, we have \( f = (a - \beta)\gamma > -1 \) and $c = (n + 1 - \beta)\gamma - (n + 1) < 0$. By Theorem 1.12 in [7], there is a constant $C_2 = C_2(n, \alpha, \beta)$ so that for all $w \in B_n$,
\[
\int_{B_n} \frac{(1 - |\eta|^2)^{(a - \beta)\gamma}}{|1 - \langle w, \eta \rangle|^{\gamma(n + 1 + a - 2\beta)\gamma}} d\nu(\eta) \leq C_2.
\]

We then have the inequality in (ii) with $C_2 = c_\alpha (1 - \delta^{2n})^{1 - 1/\gamma}(C_2)^{1/\gamma}$. Since $\delta \uparrow 1$ as $R \uparrow 1$, we see that $C_2 \to 0$ as $R \uparrow 1$. \(\blacksquare\)

With the help of Lemma 2.1, the proof of the following lemma is the same as that of Lemma 2.6 in [6].

**Lemma 2.2.** Suppose that $0 < r < R < 1$ and $\{w_j\}^\infty_{j=1} \subset B_n$ so that $E(w_{j_1}, r) \cap E(w_{j_2}, r) = \emptyset$ for all $j_1 \neq j_2$. Let $a_{j_1} A_{j_1} \in L^\infty$ be functions of norm $\leq 1$ such that $a_{j_1}(z) = 0$ for almost every $z$ in $B_n \setminus E(w_{j_1}, r)$ and $A_{j_1}(z) = 0$ for almost every $z$ in $E(w_{j_1}, R)$, $j = 1, 2, \ldots$. Then the operator \( \sum_{j \in \mathbb{N}} M_{a_j} P A_{j_1} \) is bounded on $L^2$, with norm bounded by some constant $k(R) \to 0$ when $R \to 1$.

The following proposition is a version of Proposition 2.9 in [6] for Toeplitz operators on the weighted Bergman space $A^2_\rho$. Since we have Lemma 2.2 and all the needed properties of the metric $\rho$, the proof is identical to that of Suárez.

**Proposition 2.3.** Suppose $0 < r < 1$ and $\{w_j\}^\infty_{j=1} \subset B_n$ so that $E(w_{j_1}, r) \cap E(w_{j_2}, r) = \emptyset$ for all $j_1 \neq j_2$. Let $c_{j_1}, c_{j_2}, a_{j_1}, b_{j_1}, d_{j_1}^1, \ldots, d_{j_1}^k \in L^\infty$ be functions of norm $\leq 1$ that vanish almost everywhere on $B_n \setminus E(w_{j_1}, r)$, $j = 1, 2, \ldots$. Then the operator
\[
\sum_{j \in \mathbb{N}} T_{c_{j_1}} \cdots T_{c_{j_2}} (T_{b_{j_1}}^* T_{b_{j_2}} - T_{b_{j_1}}^* T_{b_{j_2}}) T_{d_{j_1}^1} \cdots T_{d_{j_1}^k} T_{d_{j_1}^k}^*
\]
belongs to the commutator ideal $C(I(G))$, where $G$ is the closed subalgebra of $L^\infty$ generated by the functions $c^1_j, \ldots, c^1_j, a_j, b_j, d^1_j, \ldots, d^k_j$, $j = 1, 2, \ldots$.

For any multi-indices $m = (m_1, \ldots, m_n)$ and $k = (k_1, \ldots, k_n)$ in $\mathbb{N}^n$, we say that $m \preceq k$ (or $k \succeq m$) if $m_j \leq k_j$ for all $1 \leq j \leq n$. We write $m + k = (m_1 + k_1, \ldots, m_n + k_n)$ and $k - m = (k_1 - m_1, \ldots, k_n - m_n)$. It is clear that $k - m \in \mathbb{N}^n$ if $m \preceq k$. We have $|m + k| = |m| + |k|$ and $|k - m| = |k| - |m|$. Recall that the standard orthonormal basis for $A^\infty_k$ is $\{e_m : m \in \mathbb{N}^n\}$, where

$$e_m(z) = \left[ \frac{\Gamma(n + |m| + \alpha + 1)}{\Gamma(n + \alpha + 1)} \right]^{1/2} z^m, \ m \in \mathbb{N}^n, z \in \mathbb{B}_n.$$ 

It is clear that for any multi-indices $m$ and $k$, there is a positive constant $d_a(m, k)$ so that $e_m e_k = d_a(m, k) e_{m+k}$. Note that $d_a(0, k) = d_a(m, 0) = 1$ for all $m$ and $k$. Here $0$ denotes $(0, \ldots, 0)$.

A function $f$ on $\mathbb{B}_n$ is called a radial function if there is a function $\tilde{f}$ defined on $[0, 1)$ so that $f(z) = \tilde{f}(|z|)$ for all $z \in \mathbb{B}_n$. For such an $f$ and any real number $s \geq 0$, put

$$\omega_a(f, s) = \frac{\Gamma(n + s + \alpha + 1)}{\Gamma(n + \alpha + 1)} \frac{1}{\Gamma(n + s)} \int_0^1 r^{n+s-1}(1 - r)^a \tilde{f}(r^{1/2}) \, dr.$$ 

If $f(z) = \chi_{E(0, \delta)}(z) = \chi_{[0, \delta]}(|z|)$ for some $0 < \delta < 1$ then for any $s \geq 0$,

$$\omega_a(\chi_{E(0, \delta)}, s) = \frac{\Gamma(n + s + \alpha + 1)}{\Gamma(n + \alpha + 1)} \frac{1}{\Gamma(n + s)} \int_0^1 r^{n+s-1}(1 - r)^a \chi_{[0, \delta]}(r^{1/2}) \, dr$$

$$= \frac{\Gamma(n + s + \alpha + 1)}{\Gamma(n + \alpha + 1)} \frac{\delta^2}{\Gamma(n + s)} \int_0^{\delta^2} (1 - r)^a \, dr.$$ 

Since $\min\{1, (1 - \delta^2)^a\} \leq (1 - r)^a \leq \max\{1, (1 - \delta^2)^a\}$ for all $0 \leq r \leq \delta^2$, we have

$$\min\{1, (1 - \delta^2)^a\} \leq \omega_a(\chi_{E(0, \delta)}, s) \leq \omega_a(\chi_{E(0, \delta)}, s) \frac{\Gamma(n + s + \alpha + 1)}{\Gamma(n + s + \alpha + 1) \delta^2(n+s)} \leq \max\{1, (1 - \delta^2)^a\}.$$ 

This then implies that for any $0 < r < R < 1$, we have

$$\lim_{s \to \infty} \omega_a(\chi_{E(0,r)}, s) = 0.$$ 

**Lemma 2.4.** Let $f$ be a bounded radial function on $\mathbb{B}_n$. Then for any multi-indices $m$ and $k$ we have

$$P_a(f e_m e_k) = \begin{cases} d_a(k, m - k) \omega_a(f, |m|) e_{m-k} & \text{if } k \preceq m, \\ 0 & \text{otherwise.} \end{cases}$$
Proof. For any multi-index $l$, consider
\[
\langle P_n(f e_m \bar{e}_k), e_l \rangle = \langle f e_m, e_{k+l} \rangle,
\]
\[
d_n(k, l) (f e_m, e_{k+1}) =
\]
\[
d_n(k, l) c_a \int f(z) e_m(z) \bar{e}_{k+l}(z) (1 - |z|^2) d\nu(z) \]
\[
= d_n(k, l) c_a \int \frac{2nr^{2n-1}}{2n} (1 - r^2)^{\alpha} f(r) dr \]
\[
\int \frac{1}{2} e_m(r \xi) \bar{e}_{k+l}(r \xi) d\sigma(\xi) \]
\[
= d_n(k, l) c_a \int \frac{2n r^{2n+|m|+|k|+l}}{2n} (1 - r^2)^{\alpha} f(r) dr \]
\[
\int \frac{1}{2} e_m(r \xi) \bar{e}_{k+l}(r \xi) d\sigma(\xi). \]

Thus if $m \neq k + l$, $\langle P_n(f e_m \bar{e}_k), e_l \rangle = 0$. If $m = k + l$, we have
\[
\langle P_n(f e_m \bar{e}_k), e_l \rangle =
\]
\[
d_n(k, l) c_a \frac{\Gamma(n + |m| + \alpha + 1)}{m! \Gamma(n + 1 + \alpha)} \frac{\Gamma(n) m!}{\Gamma(n + |m|)} \int 2n r^{2n+2|m|-1} (1 - r^2)^{\alpha} f(r) dr
\]

(see Lemma 1.11 in [7])
\[
= d_n(k, l) \frac{\Gamma(n + |m| + \alpha + 1)}{\Gamma(n + |m|)} \Gamma(n + 1 + \alpha) \int 2n r^{2n+2|m|-1} (1 - r^2)^{\alpha} f(r) dr
\]
\[
= d_n(k, l) \frac{\Gamma(n + |m| + \alpha + 1)}{\Gamma(n + |m|)} \Gamma(n + 1 + \alpha) \int r^{n+|m|+1} (1 - r^2)^{\alpha} f(r) dr
\]
\[
= d_n(k, l) \omega_n(f, |m|).
\]

Therefore,
\[
P_n(f e_m \bar{e}_k) =
\]
\[
\begin{cases} 
  d_n(k, m - k) \omega_n(f, |m|) e_{m-k} & \text{if } k \leq m, \\
  0 & \text{otherwise}. 
\end{cases}
\]

Remark 2.5. Lemma 2.4 implies that for a radial function $f \in L^\infty$, we have $T f e_m = \omega_n(f, |m|) e_m$ for all multi-indices $m$. Therefore, the Toeplitz operator $T_f$ is diagonal with respect to the standard orthonormal basis. In fact, $T_f = \sum_{m \in \mathbb{N}^n} \omega_n(f, |m|) e_m \otimes e_m$. Here $(e_m \otimes e_m) h = \langle h, e_m \rangle e_m$ for all $h \in A^2_\alpha$.

Lemma 2.6. Let $\{s_m : m \in \mathbb{N}^n\}$ be a bounded set of strictly positive real numbers. Let $0 < r < R < 1$. Then for any $\delta > 0$, there is a number $\lambda > 0$ depending on $\delta, r, R$...
and \( \left\{ \frac{\omega_n(\chi_E(0,r), |m|)}{s_m} : m \in \mathbb{N}^n \right\} \) so that

\[
T_{\chi_E(0,r)} \leq \lambda \sum_{m \in \mathbb{N}^n} s_m e_m \otimes e_m + \delta T_{\chi_E(0,R)}.
\]

**Proof.** As in the remark preceding the lemma,

\[
T_{\chi_E(0,r)} = \sum_{m \in \mathbb{N}^n} \omega_n(\chi_E(0,r), |m|) e_m \otimes e_m,
\]

and

\[
T_{\chi_E(0,R)} = \sum_{m \in \mathbb{N}^n} \omega_n(\chi_E(0,R), |m|) e_m \otimes e_m.
\]

Let \( \delta > 0 \) be given. We need to find a \( \lambda > 0 \) so that

\[
\omega_n(\chi_E(0,r), |m|) \leq \lambda s_m + \delta \omega_n(\chi_E(0,R), |m|)
\]

for all multi-indices \( m \). By (2.1), there is an integer \( M \geq 1 \) so that

\[
\omega_n(\chi_E(0,r), |m|) \leq \delta \omega_n(\chi_E(0,R), |m|)
\]

for all multi-indices \( m \) with \( |m| \geq M \). Put

\[
\lambda = \max \left\{ \frac{\omega_n(\chi_E(0,r), |m|)}{s_m} : |m| \leq M \right\}.
\]

We then have the required inequality. \( \square \)

For any \( 1 \leq j \leq n \), let \( \delta_j = (\delta_{1j}, \ldots, \delta_{nj}) \), where \( \delta_{ij} = 1 \) if \( i = j \) and 0 otherwise.

**Proposition 2.7.** Let \( \{p_s\}_{s=1}^{\infty} \) be a sequence of distinct positive integers so that \( \sum_{s=1}^{\infty} p_s^{-1} = \infty \). Let \( f \in L^\infty \) be a radial function which is not identically zero in \( \mathbb{B}_n \). Let \( G \) be a self-adjoint subset of \( L^\infty \) so that for any integer \( s \geq 1 \) and integer \( 1 \leq j \leq n \), there is a multi-index \( k \in \mathbb{N}^n \) with \( |k| = p_s \) and \( f e_k \bar{e}_k + \delta_{ij} \in G \) (hence, \( f e_k \bar{e}_k + \delta_{ij} \in G \) as well). Then \( \mathcal{T}(G) \) is irreducible.

**Proof.** Suppose \( Q \) is an operator on \( A_2^n \) so that \( TQ = QT \) for all \( T \in \mathcal{T}(G) \). Let \( m \) be a multi-index in \( \mathbb{N}^n \). Since \( f \) is not identically zero and \( \sum_{s=1}^{\infty} p_s^{-1} \) diverges, there is an integer \( s \geq 1 \) so that \( \omega_n(f, p_s + 1 + |m|) \neq 0 \). This is a consequence of Muntz-Szasz’s Theorem (Theorem 15.26 in [5]). Now let \( 1 \leq j \leq n \). By the assumption about \( G \), there is a multi-index \( k \in \mathbb{N}^n \) so that \( |k| = p_s \) and \( f e_k \bar{e}_k + \delta_{ij} \in G \). We then have \( T_{f e_k \bar{e}_k + \delta_{ij}} Q = QT_{f e_k \bar{e}_k + \delta_{ij}} \).
Let $\varphi = Q_1 \in A^2_\mathbb{R}$. From Lemma 2.4, $T_{f_0}\overline{Q}_k \phi_{\delta_j} \varphi = \phi_{\delta_j} = 0$, so $T_{f_0}\overline{Q}_k \varphi = 0$. Using this together with the identities

\begin{equation}
T_{f_0}\overline{Q}_k \varphi = \frac{d}{d\alpha}(k + \delta_j, m)\varphi_{\alpha}(T_{f_0}\overline{Q}_k + m) = d_{\alpha}(k + \delta_j, m)\varphi_{\alpha}(T_{f_0}\overline{Q}_k + m) = d_{\alpha}(k + \delta_j, m)\varphi_{\alpha}(T_{f_0}\overline{Q}_k + m)
\end{equation}

we get,

\begin{align*}
0 &= \langle T_{f_0}\overline{Q}_k \varphi, e_n \rangle \\
&= \langle \varphi, T_{f_0}\overline{Q}_k e_n \rangle \\
&= \langle \varphi, d_{\alpha}(T_{f_0}\overline{Q}_k + m) \rangle \\
&= d_{\alpha}(k + \delta_j, m)\varphi_{\alpha}(f, p_0 + 1 + |m|)\varphi_{\alpha}(e_n + m) \rangle.
\end{align*}

From this we conclude that $\langle \varphi, e_n \rangle = 0$. Since this is true for all multi-indices $m$ and any $1 \leq j \leq n$, it follows that $\varphi$ is a constant function. So there is a complex number $c$ so that $\varphi(z) = c$ for all $z \in \mathbb{B}_n$.

We next show that $Q\alpha = ce_n$ for all multi-indices $m \in \mathbb{N}^n$ by induction on $|m|$. Since $Q_1 = c$ and $1 = e_{(0, \ldots, 0)}$, we have $Qe_m = ce_m$ for $|m| = 0$. Now suppose this holds true for all multi-indices $m$ with $|m| = l \geq 0$. Let $m'$ be a multi-index with $|m'| = l + 1$. Then there is a multi-index $m$ with $|m| = l$ and an integer $1 \leq j \leq n$ so that $m' = m + \delta_j$. Choose $s \geq 1$ so that $\omega_k(T_{f_0}\overline{Q}_k + m' + 1 + |m|) \neq 0$. Then there is a multi-index $k$ so that $|k| = p_0$ and $T_{f_0}\overline{Q}_k e_n = G$. Since $T_{f_0}\overline{Q}_k = T_{f_0}\overline{Q}_k e_n$, we have

\begin{equation}
Q\alpha(T_{f_0}\overline{Q}_k e_n) = QT_{f_0}\overline{Q}_k e_n = T_{f_0}\overline{Q}_k e_n = cP_n(T_{f_0}\overline{Q}_k + m) = ce_n.
\end{equation}

By (2.2) and the fact that $d_{\alpha}(k + \delta_j, m)\varphi_{\alpha}(T_{f_0}\overline{Q}_k + m)\varphi_{\alpha}(f, p_0 + 1 + |m|) \neq 0$, equation (2.3) implies $Qe_{m'} = ce_{m'}$.

Thus, $Q = cI$, where $I$ is the identity operator on $A^2_\mathbb{R}$. Since the commutant of $\mathcal{T}(G)$ consists of only multiples of the identity operator, we conclude that $\mathcal{T}(G)$ is irreducible.

**Remark 2.8.** Let $G_f = fC(\mathbb{B}_n) = \{fg : g \in C(\mathbb{B}_n)\}$ where $f \in L^\infty$ is a radial function which is not identically zero in $\mathbb{B}_n$. Proposition 2.7 then implies that $\mathcal{T}(G_f)$ is irreducible. Since $\mathcal{T}(G_f)$ clearly contains a nonzero compact operator (for example, $T_{f_0}$ for an appropriate $g \in C(\mathbb{B}_n)$), we conclude that $K \subset \mathcal{T}(G_f)$. If it happens that $T_f$ is compact then since $T_{f_0} - T_f$ is compact for any $g \in C(\mathbb{B}_n)$ (see [1]), we see that $T_{f_0}$ is compact for all $g \in C(\mathbb{B}_n)$. Hence in this case, $\mathcal{T}(G_f) = K$. 
For each $z \in \mathbb{B}_n$, the formula $U_z(f) = (f \circ \varphi_z)k_z$, $f \in L^2_\alpha$, defines a bounded operator on $L^2_\alpha$. It is easy to verify that $U_z$ is a self-adjoint unitary operator with $A^2_\alpha$ as a reducing subspace and $U_zT_zU_z = T_{f_z}\varphi_z$ on $A^2_\alpha$ for all $z \in \mathbb{B}_n$ and all $f \in L^\infty$. See, for example, Lemmas 7 and 8 in [3].

**Remark 2.9.** Suppose $f = h \circ \varphi_z$ for some radial function $h \in L^\infty$ which is not identically zero and $z \in \mathbb{B}_n$. Let $G_f = fC(\mathbb{E}_n)$ and $G_h = hC(\mathbb{E}_n)$ then $G_f = \{g \circ \varphi_z : g \in G_h\}$. Since $T_{\varphi_z} = U_zT_zU_z$ for any $g \in G_h$, we see that $\mathcal{T}(G_f) = U_z\mathcal{T}(G_h)U_z$. Therefore, all the statements in Remark 2.8 remain true for this $f$.

3. PROOF OF THE MAIN THEOREM

We begin this section by the following lemma which implies that for all $\epsilon \in (0, 1)$, there is a continuous function $g_\epsilon$ supported in the set $\{z \in \mathbb{B}_n : |z| < \epsilon\}$ so that $[T_{\varphi_z}, T_{\varphi_{\epsilon}}]$ is a diagonal operator (with respect to the standard orthonormal basis of $A^2_\alpha$) which has only nonzero eigenvalues.

**Lemma 3.1.** Let $g : [0, \infty) \rightarrow [0, \infty)$ be a measurable function with $\|g\|_\infty > 0$ and $g(t) = 0$ for all $t \geq 1$. For any $0 < \epsilon < 1$ and any multi-index $m_0 \in \mathbb{N}^n$ with $|m_0| > 1$, let $g_\epsilon(z) = \frac{g(|z|\epsilon^{-1})}{|z|^n}$ and $f_{\epsilon,m_0}(z) = e_{m_0}(z)g_\epsilon(z)$, for all $z \in \mathbb{B}_n$. Then the operator $T_{\epsilon,m_0} = [T_{f_{\epsilon,m_0}}, T_{\varphi_{\epsilon}}]$ is diagonal with respect to the standard orthonormal basis. Furthermore, for each fixed $m_0$, for all but at most countably many $\epsilon$, $T_{\epsilon,m_0}e_m \neq 0$ for all $m \in \mathbb{N}^n$.

**Proof.** For any multi-index $m$, from Lemma 2.4 we have

$$T_{f_{\epsilon,m_0}}e_m = P_\alpha(e_{m_0}g_\epsilon e_m)$$

$$= P_\alpha(e_{m_0}g_\epsilon e_{m_0+m})$$

$$= d_\alpha(m, m_0)P_\alpha(g_\epsilon e_{m_0+m})$$

$$= d_\alpha(m, m_0)\omega_\alpha(g_\epsilon, |m_0 + m|)e_{m_0+m}$$

$$= d_\alpha(m, m_0)\omega_\alpha(g_\epsilon, |m_0| + |m|)e_{m_0+m},$$

and

$$T_{\varphi_{\epsilon}}e_m = P_\alpha(g_\epsilon e_{m_0}e_m)$$

$$= P_\alpha(g_\epsilon e_{m_0}e_m)$$

$$= \begin{cases} d_\alpha(m_0, m - m_0)\omega_\alpha(g_\epsilon, |m|)e_{m-m_0}, & \text{if } m_0 < m, \\ 0 & \text{otherwise.} \end{cases}$$
Therefore,
\[(T_{\epsilon,m_0} T_{\epsilon,m_0}) e_m = T_{\epsilon,m_0} (d_a(m, m_0) \omega_a(\epsilon, m_0) + |m|) e_{m \pm m}) \]
\[= d_a^2(m_0, m) \omega_a^2(\epsilon, m_0) + |m|) e_{m}, \]
\[(T_{\epsilon,m_0} T_{\epsilon,m_0}) e_m = \begin{cases} 
  d_a^2(m_0, m - m_0) \omega_a^2(\epsilon, m_0) |m|) e_{m} & \text{if } m_0 \leq m, \\
  0 & \text{otherwise}. 
\end{cases} \]

It then follows that $T_{\epsilon,m_0} = [T_{\epsilon,m_0}, T_{\epsilon,m_0}]$ is a diagonal operator with respect to the standard orthonormal basis. We have $T_{\epsilon,m_0} = \sum_{m \in \mathbb{N}} s_m(\epsilon, m_0) e_m$, where $s_m(\epsilon, m_0) = d_a^2(m_0, m - m_0) \omega_a^2(\epsilon, m_0) - d_a^2(m_0, m) \omega_a^2(\epsilon, m_0) + |m|$ if $m_0 \leq m$ and $s_m(\epsilon, m_0) = -d_a^2(m_0, m) \omega_a^2(\epsilon, m_0) + |m|$ if $m_0 \leq m$.

For any real number $s \geq 0$,
\[\omega_a(\epsilon, s) = \frac{\Gamma(n + s + \alpha + 1)}{\Gamma(n + s)} \int_0^1 (1 - r)^a \tilde{\tilde{g}} r^{1/2} e^{-1} dr \]
\[= \frac{\Gamma(n + s + \alpha + 1)}{\Gamma(n + s)} e^{2n + 2s} \int_0^1 u^{n + s - 1} (1 - e^2u)^a \tilde{\tilde{g}} (u^{1/2}) du.\]
(by the change-of-variable $u = e^{-2r}$).

Now suppose $|m_0| \geq 1$. By assumption about $\tilde{\tilde{g}}$, we have $s_m(\epsilon, m_0) \neq 0$ for all $0 < \epsilon < 1$ if $m \neq m_0$. Now let $m \geq m_0$. Then $s_m(\epsilon, m_0)$ can be written as
\[s_m(\epsilon, m_0) = A \int_0^1 (e^{2|m_0| - u^{m_0}|} - B)u^{n + |m| - 1} (1 - e^2u)^a \tilde{\tilde{g}} (u^{1/2}) du,\]
where $A$ and $B$ are nonzero numbers with $B$ independent of $\epsilon$.

Since the function
\[\theta(\tau) = \int_0^1 (\tau^{2|m_0| - u^{m_0}|} - B)u^{n + |m| - 1} (1 - \tau^2u)^a \tilde{\tilde{g}} (u^{1/2}) du\]
is holomorphic in the open unit disc $|\tau| < 1$ and $\theta(0) \neq 0$, the set $\{\tau : |\tau| < 1 \text{ and } \theta(\tau) = 0\}$ is at most countable. So the set $\{\epsilon : 0 < \epsilon < 1 \text{ and } s_m(\epsilon, m_0) = 0\}$ is at most countable for each multi-index $m$. This implies that $\{\epsilon : 0 < \epsilon < 1 \text{ and } s_m(\epsilon, m_0) = 0 \text{ for some } m \in \mathbb{N}\}$ is at most countable. The conclusion of the lemma then follows. $\blacksquare$

We are now in the position of proving Theorem 1.1.

Proof of Theorem 1.1. First of all, Remark 2.8 shows that $\mathcal{K} \subset \mathcal{I}(G)$. Second, by Lemma 3.1, there is an $\epsilon > 0$ and a continuous function $\eta_\epsilon$ supported in $E(0, r)$ so that $[T_{\eta_\epsilon}, T_{\eta_\epsilon}]$ is a diagonal operator (with respect to the standard orthonormal basis) with only nonzero eigenvalues.
Fix an \( \tilde{R} \) so that \( R < \tilde{R} < 1 \). Let \( \delta > 0 \) be given. Applying Lemma 2.6, we get a number \( \lambda > 0 \) so that
\[
T_{\chi E(0, R)} \leq \lambda [T_{\eta_0, T_{\eta_1}}, T_{\chi E(0, R)}] + \delta T_{\chi E(0, R)}.
\]
For each \( j \in J \), multiplying \( U_{z_j} \) to both sides of the above inequality and using the identities \( U_{z_j}T_f U_{z_j} = T_{\phi_j \circ \varphi_j} \), \( \varphi_j(E(0, R)) = E(z_j, R) \) and \( \varphi_j(E(0, \tilde{R})) = E(z_j, \tilde{R}) \), we get
\[
T_{\chi E(z_j, R)} \leq \lambda U_{z_j} [T_{\eta_0, T_{\eta_1}}, T_{\chi E(z_j, R)}] + \delta T_{\chi E(z_j, R)} \leq \lambda [T_{\eta_0 \circ \varphi_j, T_{\eta_1 \circ \varphi_j}}, T_{\chi E(z_j, R)}] + \delta T_{\chi E(z_j, R)}.
\]
It then follows that
\[
T_{\chi W} \leq \sum_{j \in J} T_{\chi E(z_j, R)} \leq \lambda \sum_{j \in J} [T_{\eta_0 \circ \varphi_j, T_{\eta_1 \circ \varphi_j}}, T_{\chi E(z_j, R)}] + \delta \sum_{j \in J} T_{\chi E(z_j, R)}.
\]
Note that all the sums above are taken in the strong operator topology of \( \mathfrak{B}(L^2_0) \).

Now since the set \( \{z_j : j \in J\} \) is separated, by Lemma 2.3 in [2] we can decompose \( J = J_1 \cup \cdots \cup J_M \) where \( E(z_{j_1}, \tilde{R}) \cap E(z_{j_2}, \tilde{R}) = \emptyset \) for all \( j_1 \neq j_2 \) in \( J_s \), \( 1 \leq s \leq M \). Then
\[
\sum_{j \in J} T_{\chi E(z_j, R)} = \sum_{s=1}^M \sum_{j \in J_s} T_{\chi E(z_j, R)} \leq \sum_{s=1}^M T_{\chi \eta_s} = M.
\]

On the other hand, Proposition 2.3 implies that the operators
\[
A_s = \sum_{j \in J_s} [T_{\eta_0 \circ \varphi_j, T_{\eta_1 \circ \varphi_j}}, T_{\chi E(z_j, R)}] - \sum_{s=1}^M A_s \text{ also belongs to } \mathfrak{C}(G).
\]
Now inequality (3.1) becomes \( T_{\chi W} \leq \lambda A + \delta M \).

Let \( \pi \) denote the canonical quotient map from \( \mathfrak{S}(G) \) onto the quotient algebra \( \mathfrak{S}(G)/\mathfrak{C}(G) \). Since \( \pi(A) = 0 \), we then have \( 0 \leq \pi(T_{\chi W}) \leq \delta M \) for arbitrary \( \delta > 0 \). This implies \( \pi(T_{\chi W}) = 0 \). For any function \( f \in G \) with \( f \geq 0 \), we have \( 0 \leq f \leq \|f\|_{\infty} T_{\chi W} \). This gives \( 0 \leq T_{f} \leq \|f\|_{\infty} T_{\chi W} \) and hence, \( \pi(T_f) = 0 \). For arbitrary function \( f \in G \), since \( f \) can be written as a linear combination of positive functions in \( G \), we conclude that \( \pi(T_f) = 0 \). So \( \mathfrak{C}(G) = \mathfrak{S}(G) \).

Now if \( J \) is finite then for any \( f \) in \( G \), \( f \) vanishes a.e. on the complement of a compact subset of \( \mathbb{B}_M \), so \( T_f \) is compact. This implies that \( \mathfrak{S}(G) \subseteq \mathcal{K} \). Hence \( \mathcal{K} = \mathfrak{C}(G) = \mathfrak{S}(G) \).

Finally, suppose \( J = \mathbb{N} \). We will show that \( T_{\chi W} \) is not compact and this implies that \( \mathcal{K} \subseteq \mathfrak{S}(G) \). Since the sequence \( \{z_j\}_{j=1}^{\infty} \) is separated, we have \( |z_j| \to 1 \) as \( j \to \infty \). Hence \( k_{z_j} \to 0 \) weakly as \( j \to \infty \). But for each \( j \), since \( k_{z_j} = U_{z_j}1 \), we
have
\[
\langle T_{XW}k_z, k_z \rangle \geq \langle T_{X_E(0,1)}k_z, k_z \rangle = \langle U_z T_{X_E(0,1)} U_z 1, U_z 1 \rangle = \langle T_{X_E(0,1)} 1, 1 \rangle.
\]
So \(\|T_{XW}k_z\| \geq \langle T_{X_E(0,1)} 1, 1 \rangle\) for all \(j \geq 1\). Hence \(T_{XW}\) is not compact. Thus in this case \(\mathcal{K} \subseteq \mathcal{C}\mathfrak{T}(G) = \mathcal{I}(G)\).

Acknowledgements. This work was completed when the author was attending the Fall 2007 Thematic Program on Operator Algebras at the Fields Institute for Research in Mathematical Sciences. The author wishes to thank the referee for many valuable suggestions.

REFERENCES