FINITE RANK TOEPLITZ OPERATORS

TRIEU LE

ABSTRACT. In this note we offer a simplified proof of a theorem by Borichev and Rozenblum \cite{2} on finite rank Toeplitz operators whose symbols may have unbounded supports.

For the background on the problem, the reader is referred to \cite{1, 2}. It is now a well-known result of D. Luecking \cite{4} that if $\mu$ is a complex Borel measure with a compact support such that the functional

$$ (T_\mu p)(q) = \int_C p\overline{q} \, d\mu \quad \text{for } p, q \text{ analytic polynomials}, $$

has finite rank, then $\mu$ is a finite combination of point masses. As a consequence, if $\varphi$ is a bounded function with a compact support and $T_\varphi$ has finite rank on the Fock space $\mathcal{F}^2$, then $\varphi \equiv 0$.

Luecking’s proof does not carry over to the case where the measure $\mu$ has an unbounded support. In fact, there are examples where $\mu \not\equiv 0$ but $T_\mu \equiv 0$. This was discovered by Grudsky and Vasilevski \cite{3}. Concrete examples were presented in \cite[Proposition 4.6]{1}.

In \cite{5}, Rozenblum obtained Luecking’s Theorem for non-compactly supported measures with certain decay restrictions at infinity. Very recently, Borichev and Rozenblum \cite{2} settled the finite rank problem, proving that if $\varphi$ is bounded and $T_\varphi$ has finite rank on $\mathcal{F}^2$, then $\varphi \equiv 0$. In this note we provide a simplification of their proof.

We first recall the following result from \cite{1}.

**Lemma 1.** Let $\varphi$ be a bounded measurable function. Suppose $f_1, \ldots, f_N$ and $g_1, \ldots, g_N$ are functions in $\mathcal{F}^2$ such that $T_\varphi = \sum_{j=1}^{N} \langle \cdot, f_j \rangle g_j$. Then the function $W(z) = \sum_{j=1}^{N} f_j(z)g_j(-z)$ and all of its partial derivatives vanish at infinity.

Furthermore, if $W \equiv 0$, then $\varphi \equiv 0$ almost everywhere.

It was shown in \cite{2} that such a function $W$ in Lemma 1 must vanish identically on $\mathbb{C}$. The main purpose of this note is to provide a simplified proof of this result.

**Theorem 2** (Borichev-Rozenblum). Let $f_1, \ldots, f_N$ and $g_1, \ldots, g_N$ be entire functions. Put

$$ F(z) = f_1(z)g_1(z) + \cdots + f_N(z)g_N(z) \quad \text{for } z \in \mathbb{C}. $$

Suppose all partial derivatives $\partial_z^k \partial_{\overline{z}}^l F$ with $0 \leq k, l \leq N-1$ vanish at infinity. Then $F(z) = 0$ for all $z \in \mathbb{C}$. 

1
We first prove an auxiliary result. We shall think of any vector in \( \mathbb{C}^N \) as a column vector. For \( \mathbf{v}_0, \ldots, \mathbf{v}_{N-1} \) in \( \mathbb{C}^N \), we use \( \det(\mathbf{v}_0, \ldots, \mathbf{v}_{N-1}) \) to denote the determinant of the matrix whose \( j \)th column is the vector \( \mathbf{v}_j \), for each \( 0 \leq j \leq N - 1 \).

**Lemma 3.** Let \( \mathbf{v}_0, \ldots, \mathbf{v}_{N-1} \) and \( \mathbf{u}_0, \ldots, \mathbf{u}_{N-1} \) be vectors in \( \mathbb{C}^N \). Suppose there is a number \( \epsilon > 0 \) such that \( |\langle \mathbf{v}_k, \mathbf{u}_l \rangle| \leq \epsilon \) for all \( 0 \leq k, l \leq N - 1 \). Then

\[
|\det(\mathbf{v}_0, \ldots, \mathbf{v}_{N-1}) \det(\mathbf{u}_0, \ldots, \mathbf{u}_{N-1})| \leq (\epsilon \sqrt{N})^N.
\]

**Proof.** Let \( A \) denote the matrix whose columns are the vectors \( \mathbf{v}_0, \ldots, \mathbf{v}_{N-1} \) and \( B \) be the matrix whose columns are \( \mathbf{u}_0, \ldots, \mathbf{u}_{N-1} \). By assumption, the modulus of each entry of the product \( B^*A \) is at most \( \epsilon \). Hadamard’s inequality gives \( |\det(B^*A)| \leq (\epsilon \sqrt{N})^N \). Since

\[
|\det(B^*A)| = |\det(B^*) \det(A)| = |\det(B) \det(A)| = |\det(\mathbf{v}_0, \ldots, \mathbf{v}_{N-1}) \det(\mathbf{u}_0, \ldots, \mathbf{u}_{N-1})|,
\]

the conclusion of the lemma follows. \( \square \)

**Proof of Theorem 2**. For the purpose of obtaining a contradiction, suppose \( W \) were not identically zero on \( \mathbb{C} \). By combining the functions if necessary, we may assume that the functions \( f_1, \ldots, f_N \) are linearly independent and \( g_1, \ldots, g_N \) are also linearly independent, where \( N \geq 1 \).

For \( 0 \leq j \leq N - 1 \), let \( \mathbf{v}_j \) (respectively, \( \mathbf{u}_j \)) be a column vector whose components are the derivatives \( f_1^{(j)}, \ldots, f_N^{(j)} \) (respectively, \( g_1^{(j)}, \ldots, g_N^{(j)} \)). Let \( F \) (respectively, \( G \)) denote the Wronskian of the functions \( f_1, \ldots, f_N \) (respectively, \( g_1, \ldots, g_N \)). We then have \( F(z) = \det(\mathbf{v}_1(z), \ldots, \mathbf{v}_N(z)) \) and \( G(z) = \det(\mathbf{u}_1(z), \ldots, \mathbf{u}_N(z)) \).

Let \( \epsilon > 0 \) be given. By the hypothesis, there is a number \( R_\epsilon > 0 \) such that

\[
|\langle \mathbf{v}_k(z), \mathbf{u}_l(z) \rangle| = |f_1^{(k)}(z)\overline{g}_1^{(l)}(z) + \cdots + f_N^{(k)}(z)\overline{g}_N^{(l)}(z)| \leq \epsilon,
\]

for \( |z| > R_\epsilon \) and all \( 0 \leq k, l \leq N - 1 \). Using Lemma 3, we conclude that \( |F(z)G(z)| \leq (\epsilon \sqrt{N})^N \) for all such \( z \). This implies that the entire function \( F \cdot G \) vanishes at infinity. It follows that either \( F \equiv 0 \) or \( G \equiv 0 \). Without loss of generality, we may assume that \( F \equiv 0 \), which implies that the functions \( f_1, \ldots, f_N \) are linearly dependent. This gives a contraction. \( \square \)

Combining Theorem 2 and Lemma 3, we conclude

**Theorem 4** (Borichev-Rozenblum). Let \( \varphi \) be a bounded function on \( \mathbb{C} \). If \( T_\varphi \) has finite rank on \( \mathcal{F}^2 \), then \( \varphi = 0 \) almost everywhere.

**References**


Department of Mathematics and Statistics, Mail Stop 942, University of Toledo, Toledo, OH 43606

E-mail address: trieutle2@utoledo.edu